

Multiple zeta functions of Kaneko-Tsumura type and their values at positive integers

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Abstract

Recently, a new kind of multiple zeta functions $\eta(k_1, \dots, k_r; s_1, \dots, s_r)$ was introduced by Kaneko and Tsumura. This is an analytic function of complex variables s_1, \dots, s_r , while k_1, \dots, k_r are non-positive integer parameters. In this paper, we first extend this function to an analytic function $\eta(s'_1, \dots, s'_r; s_1, \dots, s_r)$ of $2r$ complex variables. Then we investigate its special values at positive integers. In particular, we prove that there are some linear relations among these η -values and the multiple zeta values $\zeta(k_1, \dots, k_r)$ of Euler-Zagier type.

1 Introduction

In a recent paper [3], M. Kaneko and H. Tsumura introduced and studied a new kind of multiple zeta functions

$$\eta(k_1, \dots, k_r; s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{\text{Li}_{k_1, \dots, k_r}(1 - e^{-t})}{1 - e^{-t}} t^{s-1} dt, \quad (1.1)$$

which is a ‘twin sibling’ of the Arakawa-Kaneko multiple zeta function [1]

$$\xi(k_1, \dots, k_r; s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{\text{Li}_{k_1, \dots, k_r}(1 - e^{-t})}{e^t - 1} t^{s-1} dt. \quad (1.2)$$

Here k_1, \dots, k_r are integers, s is a complex variable and $\text{Li}_{k_1, \dots, k_r}$ denotes the multiple polylogarithm of one variable

$$\text{Li}_{k_1, \dots, k_r}(z) := \sum_{0 < n_1 < \dots < n_r} \frac{z^{n_r}}{n_1^{k_1} \dots n_r^{k_r}}.$$

Among other things, when $r = 1$, they proved the equality

$$\eta(k; l) = \eta(l; k) \quad (1.3)$$

for nonpositive integers k, l , and experimentally observed that the same equality holds even when k and l are positive integers.

In [3, §5], Kaneko and Tsumura also considered a variant of (1.1) with r

complex variables:

$$\begin{aligned} \eta(k_1, \dots, k_r; s_1, \dots, s_r) \\ := \frac{1}{\prod_{j=1}^r \Gamma(s_j)} \int \dots \int_0^\infty \frac{\text{Li}_{k_1, \dots, k_r}(1 - e^{\sum_{\nu=1}^r t_\nu}, 1 - e^{\sum_{\nu=2}^r t_\nu}, \dots, 1 - e^{t_r})}{\prod_{j=1}^r (1 - e^{\sum_{\nu=j}^r t_\nu})} \\ \times \prod_{j=1}^r t_j^{s_j-1} dt_j, \quad (1.4) \end{aligned}$$

where

$$\text{Li}_{k_1, \dots, k_r}(z_1, \dots, z_r) := \sum_{0 < n_1 < \dots < n_r} \frac{z_1^{n_1} z_2^{n_2 - n_1} \dots z_r^{n_r - n_{r-1}}}{n_1^{k_1} \dots n_r^{k_r}} \quad (1.5)$$

is the multiple polylogarithm of r variables. For certain technical reasons, their consideration on the function (1.4) is focused on the case that k_1, \dots, k_r are nonpositive integers.

In the present paper, we extend the function (1.4) to a holomorphic function of $2r$ complex variables $\eta(s'_1, \dots, s'_r; s_1, \dots, s_r)$, which satisfies

$$\eta(s'_1, \dots, s'_r; s_1, \dots, s_r) = \eta(s_1, \dots, s_r; s'_1, \dots, s'_r). \quad (1.6)$$

When $r = 1$, it also gives an extension of the function (1.1). In particular, we obtain a proof of the equality (1.3) for arbitrary complex numbers k, l .

The second and the main purpose of this paper is to study the special values of the function $\eta(s'_1, \dots, s'_r; s_1, \dots, s_r)$ at positive integers. We prove that there exist certain linear relations among these values and the multiple zeta values

$$\zeta(k_1, \dots, k_r) := \sum_{0 < n_1 < \dots < n_r} \frac{1}{n_1^{k_1} \dots n_r^{k_r}} \quad (1.7)$$

for positive integers k_1, \dots, k_r with $k_r > 1$. For example, as special cases of our result (Theorem 4.2), we can show that

$$\eta(\underbrace{1, \dots, 1}_r; \underbrace{1, \dots, 1}_r) = \zeta(\underbrace{2, \dots, 2}_r)$$

and

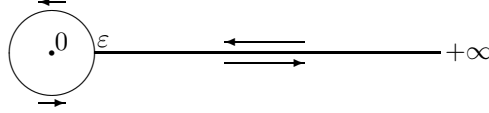
$$\eta(k; l) = \sum_{0 < a_1 \leq \dots \leq a_k = b_l \geq \dots \geq b_1 > 0} \frac{1}{a_1 \dots a_k b_1 \dots b_l}$$

(the right hand side of the latter identity can be expressed as a finite sum of multiple zeta values).

The contents of this paper is as follows. In §2, we define the function $\eta(s'_1, \dots, s'_r; s_1, \dots, s_r)$ and prove its analytic continuation to \mathbb{C}^{2r} by the classical contour integral method. In §3, basic formulas on its special values at positive integers are obtained. Some of them are used in §4, where we show certain relations of η -values with the multiple zeta values. Finally, in the appendix A, we prove a formula which expresses the values $\eta(k_1, \dots, k_r; l)$ of the function in (1.1), where k_1, \dots, k_r and l are positive integers, in terms of the multiple zeta values.

2 Definition of $\eta(s'_1, \dots, s'_r; s_1, \dots, s_r)$

Let r be a positive integer. The definition (1.5) of the multiple polylogarithm is meaningful for arbitrary complex numbers k_1, \dots, k_r and complex numbers z_1, \dots, z_r of absolute values less than 1. We begin with its analytic continuation. For a positive real number ε , denote by C_ε the contour which goes from $+\infty$ to ε along the real line, goes round counterclockwise along the circle of radius ε about the origin, and then goes back to $+\infty$ along the real line:



Lemma 2.1. *The multiple polylogarithm $\text{Li}_{\mathbf{s}}(\mathbf{z})$, where $\mathbf{s} = (s_1, \dots, s_r)$, $\mathbf{z} = (z_1, \dots, z_r) \in \mathbb{C}^r$ and $|z_i| < 1$, has the following integral expression:*

$$\text{Li}_{\mathbf{s}}(\mathbf{z}) = \prod_{j=1}^r \frac{\Gamma(1-s_j)}{2\pi i e^{\pi i s_j}} \int_{(C_\varepsilon)^r} \prod_{j=1}^r \frac{z_j u_j^{s_j-1} du_j}{e^{u_j+\dots+u_r} - z_j}. \quad (2.1)$$

Here we assume that $\varepsilon > 0$ is sufficiently small.

By (2.1), $\text{Li}_{\mathbf{s}}(\mathbf{z})$ is holomorphically continued to the region

$$(\mathbf{s}, \mathbf{z}) \in \mathbb{C}^r \times (\mathbb{C} \setminus \mathbb{R}_{\geq 1})^r.$$

Proof. First we note that

$$\begin{aligned} \prod_{j=1}^r \Gamma(s_j) \cdot \text{Li}_{\mathbf{s}}(\mathbf{z}) &= \prod_{j=1}^r \Gamma(s_j) \sum_{l_1, \dots, l_r > 0} \frac{z_1^{l_1} z_2^{l_2} \dots z_r^{l_r}}{l_1^{s_1} (l_1 + l_2)^{s_2} \dots (l_1 + \dots + l_r)^{s_r}} \\ &= \sum_{l_1, \dots, l_r > 0} \int \dots \int_0^\infty \prod_{j=1}^r e^{-(l_1 + \dots + l_j) u_j} z_j^{l_j} u_j^{s_j-1} du_j \\ &= \int \dots \int_0^\infty \prod_{j=1}^r \frac{z_j u_j^{s_j-1} du_j}{e^{u_j + \dots + u_r} - z_j}, \end{aligned}$$

that is,

$$\text{Li}_{\mathbf{s}}(\mathbf{z}) = \prod_{j=1}^r \frac{1}{\Gamma(s_j)} \int \dots \int_0^\infty \prod_{j=1}^r \frac{z_j u_j^{s_j-1} du_j}{e^{u_j + \dots + u_r} - z_j}. \quad (2.2)$$

This gives an analytic continuation to the region

$$(\mathbf{s}, \mathbf{z}) \in \{s \in \mathbb{C} \mid \Re(s) > 0\}^r \times (\mathbb{C} \setminus \mathbb{R}_{\geq 1})^r.$$

Moreover, for each $\mathbf{z} \in (\mathbb{C} \setminus \mathbb{R}_{\geq 1})^r$, there exists a neighborhood K of \mathbf{z} and $\varepsilon_0 > 0$ such that $e^{u_j + \dots + u_r} - z'_j \neq 0$ for $j = 1, \dots, r$ whenever $(z'_1, \dots, z'_r) \in K$, $0 < \varepsilon < \varepsilon_0$ and $u_1, \dots, u_r \in C_\varepsilon$. If this is the case, we have

$$\int \dots \int_0^\infty \prod_{j=1}^r \frac{z_j u_j^{s_j-1} du_j}{e^{u_j + \dots + u_r} - z_j} = \prod_{j=1}^r \frac{1}{e^{2\pi i s_j} - 1} \int_{(C_\varepsilon)^r} \prod_{j=1}^r \frac{z_j u_j^{s_j-1} du_j}{e^{u_j + \dots + u_r} - z_j}. \quad (2.3)$$

The formula (2.1) is deduced from (2.2) and (2.3) because

$$\Gamma(s_j)\Gamma(1-s_j) = \frac{\pi}{\sin \pi s_j} = \frac{2\pi i}{e^{\pi i s_j} - e^{-\pi i s_j}} = \frac{2\pi i e^{\pi i s_j}}{e^{2\pi i s_j} - 1}.$$

It is easy to see that (2.1) gives a meromorphic continuation to $\mathbb{C}^r \times (\mathbb{C} \setminus \mathbb{R}_{\geq 1})^r$. The possible poles $s_j = 1, 2, 3, \dots$, which comes from the factor $\Gamma(1-s_j)$, are removable, since we already know the holomorphy on $\Re(s_j) > 0$ from (2.2). \square

Remark 2.2. H. Tsumura pointed out to the author that the above lemma is a special case of Komori's result [4].

Definition 2.3. For $\mathbf{s} = (s_1, \dots, s_r), \mathbf{s}' = (s'_1, \dots, s'_r) \in \mathbb{C}^r$ with $\Re(s_j) > 0$, we define

$$\eta(\mathbf{s}'; \mathbf{s}) := \prod_{j=1}^r \frac{1}{\Gamma(s_j)} \int \cdots \int_0^\infty \frac{\text{Li}_{\mathbf{s}'}(1 - e^{t_1 + \cdots + t_r}, \dots, 1 - e^{t_r})}{(1 - e^{t_1 + \cdots + t_r}) \cdots (1 - e^{t_r})} \prod_{j=1}^r t_j^{s_j-1} dt_j. \quad (2.4)$$

Proposition 2.4. The function $\eta(\mathbf{s}'; \mathbf{s})$ has the following integral expression:

$$\eta(\mathbf{s}'; \mathbf{s}) = \prod_{j=1}^r \frac{\Gamma(1-s_j)\Gamma(1-s'_j)}{(2\pi i)^2 e^{\pi(s_j+s'_j)}} \int_{(C_\varepsilon)^{2r}} \prod_{j=1}^r \frac{u_j^{s'_j-1} t_j^{s_j-1} du_j dt_j}{e^{u_j + \cdots + u_r} + e^{t_j + \cdots + t_r} - 1}. \quad (2.5)$$

Proof. When $\Re(s_j), \Re(s'_j) > 0$, we substitute (2.2) (for $z_j = 1 - e^{t_j + \cdots + t_r}$) into the definition (2.4) of $\eta(\mathbf{s}'; \mathbf{s})$ to obtain

$$\eta(\mathbf{s}'; \mathbf{s}) = \prod_{j=1}^r \frac{1}{\Gamma(s_j)\Gamma(s'_j)} \int \cdots \int_0^\infty \prod_{j=1}^r \frac{u_j^{s'_j-1} t_j^{s_j-1} du_j dt_j}{e^{u_j + \cdots + u_r} + e^{t_j + \cdots + t_r} - 1}. \quad (2.6)$$

Then, by transforming the integrals on $\mathbb{R}_{>0}$ to those on the contour C_ε , we get the desired expression. \square

Corollary 2.5. $\eta(\mathbf{s}'; \mathbf{s})$ can be holomorphically continued to $\mathbb{C}^r \times \mathbb{C}^r$, and satisfies $\eta(\mathbf{s}'; \mathbf{s}) = \eta(\mathbf{s}; \mathbf{s}')$.

Remark 2.6. Recall that the Euler-Zagier multiple zeta function

$$\zeta(s_1, \dots, s_r) = \sum_{0 < n_1 < \cdots < n_r} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}$$

has the integral expression

$$\zeta(s_1, \dots, s_r) = \prod_{j=1}^r \frac{1}{\Gamma(s_j)} \int \cdots \int_0^\infty \prod_{j=1}^r \frac{u_j^{s_j-1} du_j}{e^{u_j + \cdots + u_r} - 1}.$$

This formula, together with (2.6), suggests that our function $\eta(\mathbf{s}'; \mathbf{s})$ may be regarded as a 'double multiple zeta function.' Furthermore, we may also consider a 'multiple multiple zeta function'

$$\eta(\mathbf{s}_1, \dots, \mathbf{s}_l) = \prod_{i=1}^l \prod_{j=1}^r \frac{1}{\Gamma(s_{ij})} \int \cdots \int_0^\infty \prod_{j=1}^r \frac{\prod_{i=1}^l t_{ij}^{s_{ij}-1} dt_{ij}}{\sum_{i=1}^l e^{t_{ij} + \cdots + t_{ir}} - 1}$$

for $\mathbf{s}_i = (s_{i1}, \dots, s_{ir})$. In this paper, however, we don't pursue such a generalization for $l \geq 3$.

3 Special values at positive integers

From now on, we study the values of η at positive integers. In particular, we are interested in the relationship between these values and the multiple zeta values. First let us recall some basic notation on the multiple zeta values.

A finite sequence $\mathbf{k} = (k_1, \dots, k_r)$ of positive integers is called an *index*. We put

$$|\mathbf{k}| := k_1 + \dots + k_r, \quad d(\mathbf{k}) := r,$$

and call them the *weight* and the *depth* of \mathbf{k} , respectively.

An index $\mathbf{k} = (k_1, \dots, k_r)$ is called *admissible* if $k_r > 1$ (or $r = 0$, that is, \mathbf{k} is the empty index). When this is the case, the multiple zeta value $\zeta(\mathbf{k})$ is defined by the multiple series (1.7), and has the iterated integral expression

$$\zeta(\mathbf{k}) = \int_{(x_{ji}) \in \Delta(\mathbf{k})} \prod_{j=1}^r \frac{dx_{j1}}{1-x_{j1}} \frac{dx_{j2}}{x_{j2}} \dots \frac{dx_{jk_j}}{x_{jk_j}}. \quad (3.1)$$

Here $\Delta(\mathbf{k})$ is a domain of dimension $|\mathbf{k}| := k_1 + \dots + k_r$ defined by

$$\Delta(\mathbf{k}) := \left\{ (x_{ji})_{\substack{j=1, \dots, r, \\ i=1, \dots, k_j}} \left| \begin{array}{l} 0 < x_{11} < \dots < x_{1k_1} < x_{21} < \dots < x_{2k_2} \\ \dots < x_{r1} < \dots < x_{rk_r} < 1 \end{array} \right. \right\}.$$

Now we return to the study of η -values. We start with an integral expression for $\eta(\mathbf{k}; \mathbf{l})$ similar to (3.1), using domains of the form

$$\nabla(\mathbf{k}) := \left\{ (x_{ji})_{\substack{j=1, \dots, r, \\ i=1, \dots, k_j}} \left| \begin{array}{l} 1 > x_{11} > \dots > x_{1k_1} > x_{21} > \dots > x_{2k_2} \\ \dots > x_{r1} > \dots > x_{rk_r} > 0 \end{array} \right. \right\}.$$

Proposition 3.1. *For any indices $\mathbf{k} = (k_1, \dots, k_r), \mathbf{l} = (l_1, \dots, l_r)$ of depth $r > 0$, we have*

$$\eta(\mathbf{k}; \mathbf{l}) = \int_{\substack{(x_{ji}) \in \nabla(\mathbf{k}) \\ (y_{ji}) \in \nabla(\mathbf{l})}} \prod_{j=1}^r \left\{ \frac{dx_{j1}}{1-x_{j1}y_{j1}} \frac{dx_{j2}}{1-x_{j2}} \dots \frac{dx_{jk_j}}{1-x_{jk_j}} \right. \\ \left. \times \frac{dy_{j2}}{1-y_{j2}} \dots \frac{dy_{jl_j}}{1-y_{jl_j}} \right\}. \quad (3.2)$$

Proof. Since all variables are positive, we can use the integral (2.6):

$$\eta(\mathbf{k}; \mathbf{l}) = \prod_{j=1}^r \frac{1}{\Gamma(k_j)\Gamma(l_j)} \int \dots \int_0^\infty \prod_{j=1}^r \frac{u_j^{k_j-1} t_j^{l_j-1} du_j dt_j}{e^{u_j+\dots+u_r} + e^{t_j+\dots+t_r} - 1}.$$

Then we make the change of variables

$$x_j = 1 - e^{-(u_j+\dots+u_r)}, \quad y_j = 1 - e^{-(t_j+\dots+t_r)},$$

which leads to

$$\eta(\mathbf{k}; \mathbf{l}) = \prod_{j=1}^r \frac{1}{\Gamma(k_j)\Gamma(l_j)} \int_{\substack{1 > x_1 > \dots > x_r > x_{r+1}=0 \\ 1 > y_1 > \dots > y_r > y_{r+1}=0}} \prod_{j=1}^r \left(\log \frac{1-x_{j+1}}{1-x_j} \right)^{k_j-1} \left(\log \frac{1-y_{j+1}}{1-y_j} \right)^{l_j-1} \frac{dx_j dy_j}{1-x_j y_j}. \quad (3.3)$$

Moreover, we have

$$\begin{aligned} \frac{1}{\Gamma(k_j)} \left(\log \frac{1-x_{j+1}}{1-x_j} \right)^{k_j-1} &= \frac{1}{(k_j-1)!} \left(\int_{x_{j+1}}^{x_j} \frac{dx}{1-x} \right)^{k_j-1} \\ &= \int_{x_j > x_{j2} > \dots > x_{jk_j} > x_{j+1}} \frac{dx_{j2}}{1-x_{j2}} \dots \frac{dx_{jk_j}}{1-x_{jk_j}}, \end{aligned}$$

and a similar formula for $\frac{1}{\Gamma(l_j)} \left(\log \frac{1-y_{j+1}}{1-y_j} \right)^{l_j-1}$. If we substitute them into (3.3), we get the result (3.2) (with $x_j = x_{j1}$ and $y_j = y_{j1}$). \square

Corollary 3.2. *For indices $\mathbf{k} = (k_1, \dots, k_r)$, $\mathbf{l} = (l_1, \dots, l_r)$, we have*

$$\eta(\mathbf{k}; \mathbf{l}) = \sum_{m_i, n_i: (*)} \prod_{i=1}^{|\mathbf{k}|} \frac{1}{m_i + m_{i+1} + \dots + m_{|\mathbf{k}|}} \prod_{i=1}^{|\mathbf{l}|} \frac{1}{n_i + n_{i+1} + \dots + n_{|\mathbf{l}|}}, \quad (3.4)$$

where the summation is taken over positive integers $m_1, \dots, m_{|\mathbf{k}|}$ and $n_1, \dots, n_{|\mathbf{l}|}$ satisfying

$$\begin{aligned} m_1 = n_1, \quad m_{k_1+1} = n_{l_1+1}, \quad m_{k_1+k_2+1} = n_{l_1+l_2+1}, \dots \\ \dots, m_{k_1+\dots+k_{r-1}+1} = n_{l_1+\dots+l_{r-1}+1}. \end{aligned} \quad (*)$$

Proof. We expand all factors of the integrand of (3.2) by

$$\frac{1}{1-x_{j1}y_{j1}} = \sum_{m=1}^{\infty} x_{j1}^{m-1} y_{j1}^{m-1}, \quad \frac{1}{1-x_{ji}} = \sum_{m=1}^{\infty} x_{ji}^{m-1}, \quad \frac{1}{1-y_{ji}} = \sum_{n=1}^{\infty} y_{ji}^{n-1}.$$

Then, with a renumbering of variables, the integral becomes

$$\int_{\substack{1 > x_1 > \dots > x_{|\mathbf{k}|} > 0 \\ 1 > y_1 > \dots > y_{|\mathbf{l}|} > 0}} \sum_{m_i, n_i: (*)} \prod_{i=1}^{|\mathbf{k}|} x_i^{m_i-1} dx_i \prod_{i=1}^{|\mathbf{l}|} y_i^{n_i-1} dy_i.$$

By exchanging the integral and the summation, and by integrating repeatedly, we obtain the formula (3.4). \square

Let $\mathbf{X} = (X_1, \dots, X_r)$ and $\mathbf{Y} = (Y_1, \dots, Y_r)$ be r -tuples of indeterminates, and define the generating function for the values $\eta(\mathbf{k}; \mathbf{l})$ by

$$F_r(\mathbf{X}; \mathbf{Y}) = \sum_{\mathbf{k}, \mathbf{l} \in (\mathbb{Z}_{>0})^r} \eta(\mathbf{k}; \mathbf{l}) X_1^{k_1-1} \dots X_r^{k_r-1} Y_1^{l_1-1} \dots Y_r^{l_r-1}. \quad (3.5)$$

Proposition 3.3. *We have*

$$F_r(\mathbf{X}, \mathbf{Y}) = \int_{\substack{1 > x_1 > \dots > x_r > 0 \\ 1 > y_1 > \dots > y_r > 0}} \prod_{j=1}^r (1-x_j)^{X_{j-1}-X_j} (1-y_j)^{Y_{j-1}-Y_j} \frac{dx_j dy_j}{1-x_j y_j}, \quad (3.6)$$

where $X_0 = Y_0 = 0$.

Proof. We substitute the integral expression (3.3) of $\eta(\mathbf{k}; \mathbf{l})$ into the definition (3.5) of $F_r(\mathbf{X}; \mathbf{Y})$, and take the summation over \mathbf{k} and \mathbf{l} using

$$\begin{aligned} \sum_{k_j=1}^{\infty} \frac{1}{\Gamma(k_j)} \left(\log \frac{1-x_{j+1}}{1-x_j} \right)^{k_j-1} X_j^{k_j-1} &= \exp \left(X_j \log \frac{1-x_{j+1}}{1-x_j} \right) \\ &= \left(\frac{1-x_{j+1}}{1-x_j} \right)^{X_j} \end{aligned}$$

and similar for y_j . Then we obtain

$$F_r(\mathbf{X}, \mathbf{Y}) = \int_{\substack{1 > x_1 > \dots > x_r > x_{r+1}=0 \\ 1 > y_1 > \dots > y_r > y_{r+1}=0}} \prod_{j=1}^r \left(\frac{1-x_{j+1}}{1-x_j} \right)^{X_j} \left(\frac{1-y_{j+1}}{1-y_j} \right)^{Y_j} \frac{dx_j dy_j}{1-x_j y_j},$$

which is equal to (3.6). \square

4 Relationship with multiple zeta values

In this section, we prove some linear relations among the values $\eta(\mathbf{k}; \mathbf{l})$ and the multiple zeta values $\zeta(\mathbf{k})$. Let us consider the \mathbb{Q} -linear spaces spanned by these values:

$$\begin{aligned} \mathcal{Z}_w &:= \langle \zeta(\mathbf{k}) \mid \mathbf{k} : \text{admissible index, } |\mathbf{k}| = w \rangle_{\mathbb{Q}} \subset \mathbb{R}, \\ \mathcal{Y}_w &:= \langle \eta(\mathbf{k}; \mathbf{l}) \mid \mathbf{k}, \mathbf{l} : \text{indices, } |\mathbf{k}| + |\mathbf{l}| = w \rangle_{\mathbb{Q}} \subset \mathbb{R}. \end{aligned}$$

Zagier's famous conjecture states that $\dim_{\mathbb{Q}} \mathcal{Z}_w = d_w$, where the sequence d_w is determined by $\sum_{w=0}^{\infty} d_w x^w = \frac{1}{1-x^2-x^3}$, and the upper estimate $\dim_{\mathbb{Q}} \mathcal{Z}_w \leq d_w$ is proved [2, 5]. Basic questions on the above spaces are:

- (i) What is the sequence $\dim_{\mathbb{Q}} \mathcal{Y}_w$? Can we make a precise conjecture as in the case of \mathcal{Z}_w ?
- (ii) How large is the intersection $\mathcal{Y}_w \cap \mathcal{Z}_w$? Does the inclusion $\mathcal{Y}_w \subset \mathcal{Z}_w$ or $\mathcal{Y}_w \supset \mathcal{Z}_w$ hold?

To investigate these questions, an important problem is to find a reasonably fast method to compute $\eta(\mathbf{k}; \mathbf{l})$ numerically. Unfortunately, the author does not have any such computational method.

In the following, we show some results related to the question (ii). First let us introduce some notation on indices.

Definition 4.1. (i) For an index $\mathbf{k} = (k_1, \dots, k_r)$ of weight k , put

$$J(\mathbf{k}) := \{k_1, k_1 + k_2, \dots, k_1 + \dots + k_{r-1}\} \subset \{1, 2, \dots, k-1\}.$$

- (ii) We say that \mathbf{k} is a *refinement* of \mathbf{k}' , and denote $\mathbf{k} \succeq \mathbf{k}'$, if $|\mathbf{k}| = |\mathbf{k}'|$ and $J(\mathbf{k}) \supset J(\mathbf{k}')$.
- (iii) For a formal linear combination $\alpha = \sum_{\mathbf{k}} a_{\mathbf{k}} \mathbf{k}$ of (finitely many) admissible indices, we linearly extend the function ζ , i.e., set $\zeta(\alpha) = \sum_{\mathbf{k}} a_{\mathbf{k}} \zeta(\mathbf{k})$. This 'linear extension' principle applies to operations below.

- (iv) For an index \mathbf{k} , we denote by \mathbf{k}^* the formal sum $\sum_{\mathbf{k}' \preceq \mathbf{k}} \mathbf{k}'$ of all indices \mathbf{k}' of which \mathbf{k} is a refinement.
- (v) For indices \mathbf{k} and \mathbf{l} , we denote $\mathbf{k} * \mathbf{l}$ the *harmonic product* of \mathbf{k} and \mathbf{l} . It is a formal sum of indices defined inductively by

$$\begin{aligned} \emptyset * \mathbf{k} &= \mathbf{k} * \emptyset = \mathbf{k}, \\ (k_1, \dots, k_r) * (l_1, \dots, l_s) &= ((k_1, \dots, k_{r-1}) * (l_1, \dots, l_s), k_r) \\ &\quad + ((k_1, \dots, k_r) * (l_1, \dots, l_{s-1}), l_s) \\ &\quad + ((k_1, \dots, k_{r-1}) * (l_1, \dots, l_{s-1}), k_r + l_s), \end{aligned}$$

where \emptyset denotes the unique index of depth 0.

- (vi) For indices $\mathbf{k} = (k_1, \dots, k_r)$ and $\mathbf{l} = (l_1, \dots, l_s)$ with $r, s > 0$, we set

$$\mathbf{k} \otimes \mathbf{l} := ((k_1, \dots, k_{r-1}) * (l_1, \dots, l_{s-1}), k_r + l_s).$$

Theorem 4.2. Denote by $I(k, r)$ the set of all indices of weight k and depth r . Then we have, for any positive integers k, l and r ,

$$\sum_{\mathbf{k} \in I(k, r)} \sum_{\mathbf{l} \in I(l, r)} \eta(\mathbf{k}; \mathbf{l}) = \zeta(\underbrace{(2, \dots, 2)}_r \otimes \underbrace{(1, \dots, 1, 0)}_{k-r}^* \otimes \underbrace{(1, \dots, 1, 0)}_{l-r}^*). \quad (4.1)$$

In particular, this gives a nonzero element of $\mathcal{Y}_{k+l} \cap \mathcal{Z}_{k+l}$.

Proof. First note that, while $\underbrace{(1, \dots, 1, 0)}_{k-r}$ and $\underbrace{(1, \dots, 1, 0)}_{l-r}$ in the right hand side are not indices because they contain 0, the formal operations work well and produce a formal sum of admissible indices of weight $k + l$. Hence the right hand side belongs to \mathcal{Z}_{k+l} . Explicitly, it is given by the multiple series

$$\sum_{\substack{m_1 > \dots > m_r > 0 \\ 0 < a_1 \leq \dots \leq a_{k-r} \leq m_1 \\ 0 < b_1 \leq \dots \leq b_{l-r} \leq m_1}} \frac{1}{m_1^2 \dots m_r^2 a_1 \dots a_{k-r} b_1 \dots b_{l-r}}. \quad (4.2)$$

Let us compute the left hand side of (4.1). Since

$$F_r(X, \dots, X; Y, \dots, Y) = \sum_{k, l \geq r} \left(\sum_{\mathbf{k} \in I(k, r), \mathbf{l} \in I(l, r)} \eta(\mathbf{k}; \mathbf{l}) \right) X^{k-r} Y^{l-r},$$

it is sufficient to compute this generating function. By (3.6), we have

$$F_r(X, \dots, X; Y, \dots, Y) = \int_{\substack{1 > x_1 > \dots > x_r > 0 \\ 1 > y_1 > \dots > y_r > 0}} (1 - x_1)^{-X} (1 - y_1)^{-Y} \prod_{j=1}^r \frac{dx_j dy_j}{1 - x_j y_j}.$$

Using the expansion $\frac{1}{1 - x_j y_j} = \sum_{n_j=1}^{\infty} (x_j y_j)^{n_j}$ and integrating with respect to $x_r, y_r, \dots, x_2, y_2$, it can be rewritten as

$$\sum_{m_1 > \dots > m_r > 0} \frac{1}{m_2^2 \dots m_r^2} \int_0^1 (1 - x)^{-X} x^{m_1-1} dx \int_0^1 (1 - y)^{-Y} y^{m_1-1} dy.$$

Then we note that

$$\begin{aligned}
\int_0^1 (1-x)^{-X} x^{m-1} dx &= B(1-X, m) = \frac{\Gamma(1-X)\Gamma(m)}{\Gamma(1-X+m)} \\
&= \frac{(m-1)!}{(1-X)(2-X)\cdots(m-X)} \\
&= \frac{1}{m} \prod_{a=1}^m \left(1 - \frac{X}{a}\right)^{-1} = \frac{1}{m} \prod_{a=1}^m \sum_{n=0}^{\infty} \frac{X^n}{a^n} \\
&= \frac{1}{m} \sum_{k=1}^{\infty} X^k \sum_{0 < a_1 \leq \cdots \leq a_k \leq m} \frac{1}{a_1 \cdots a_k},
\end{aligned}$$

which leads to

$$\begin{aligned}
&F_r(X, \dots, X; Y, \dots, Y) \\
&= \sum_{k, l \geq 0} X^k Y^l \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^2 \cdots m_r^2} \sum_{\substack{0 < a_1 \leq \cdots \leq a_k \leq m_1 \\ 0 < b_1 \leq \cdots \leq b_l \leq m_1}} \frac{1}{a_1 \cdots a_k b_1 \cdots b_l}.
\end{aligned}$$

Hence the coefficient of $X^{k-l} Y^{l-r}$ coincides with the series (4.2), and the result follows. \square

Corollary 4.3. $\mathcal{Y}_w \cap \mathcal{Z}_w \neq 0$ for any $w \geq 2$.

Example 4.4. (i) When $r = 1$, (4.1) says that

$$\begin{aligned}
\eta(k; l) &= \sum_{\substack{m > 0 \\ 0 < a_1 \leq \cdots \leq a_k \leq m \\ 0 < b_1 \leq \cdots \leq b_l \leq m}} \frac{1}{m^2 a_1 \cdots a_{k-1} b_1 \cdots b_{l-1}} \\
&= \zeta(\underbrace{(1, \dots, 1)}_k^* \circledast \underbrace{(1, \dots, 1)}_l^*)
\end{aligned} \tag{4.3}$$

for any $k, l > 0$. For example,

$$\begin{aligned}
\eta(1; 1) &= \zeta(2), \\
\eta(2; 1) &= \zeta(1, 2) + \zeta(3), \\
\eta(3; 1) &= \zeta(1, 1, 2) + \zeta(2, 2) + \zeta(1, 3) + \zeta(4), \\
\eta(2; 2) &= 2\zeta(1, 1, 2) + \zeta(2, 2) + 2\zeta(1, 3) + \zeta(4).
\end{aligned}$$

(ii) When $r = k = l$, (4.1) says that

$$\eta(\underbrace{1, \dots, 1}_r; \underbrace{1, \dots, 1}_r) = \zeta(\underbrace{2, \dots, 2}_r). \tag{4.4}$$

Remark 4.5. From the above example, we see that $\mathcal{Y}_w \subset \mathcal{Z}_w$ for $w = 2, 3, 4$ (in fact, these are equal since \mathcal{Z}_w is 1-dimensional for $w = 2, 3, 4$).

For $w = 5$, \mathcal{Y}_5 is generated by $\eta(4; 1)$, $\eta(3; 2)$, $\eta(2, 1; 1, 1)$ and $\eta(1, 2; 1, 1)$. Using (4.1), we see that

$$\eta(4; 1), \eta(3; 2), \eta(2, 1; 1, 1) + \eta(1, 2; 1, 1) \in \mathcal{Z}_5.$$

In fact, we also have $\eta(1, 2; 1, 1) \in \mathcal{Z}_5$ by Theorem 4.6 below, hence $\mathcal{Y}_5 \subset \mathcal{Z}_5$.

At present, the author does not know whether $\mathcal{Y}_6 \subset \mathcal{Z}_6$ or not.

Theorem 4.6. For integers $k, l > 0$, we have $\eta(1, k; 1, l) \in \mathcal{Z}_{k+l+2}$.

Proof. Starting from (3.2), we have

$$\begin{aligned} \eta(1, k; 1, l) &= \int_{\substack{1 > x_0 > \dots > x_k > 0 \\ 1 > y_0 > \dots > y_l > 0}} \frac{dx_0 dy_0}{1 - x_0 y_0} \frac{dx_1 dy_1}{1 - x_1 y_1} \prod_{j=2}^k \frac{dx_j}{1 - x_j} \prod_{j=2}^l \frac{dy_j}{1 - y_j} \\ &= \sum_{n_0, n_1 > 0} \int_{1 > x_0 > \dots > x_k > 0} x_0^{n_0-1} dx_0 x_1^{n_1-1} dx_1 \prod_{j=2}^k \frac{dx_j}{1 - x_j} \\ &\quad \times \int_{1 > y_0 > \dots > y_l > 0} y_0^{n_0-1} dy_0 y_1^{n_1-1} dy_1 \prod_{j=2}^l \frac{dy_j}{1 - y_j}. \end{aligned}$$

Now we compute the iterated integrals as

$$\begin{aligned} &\int_{1 > x_0 > \dots > x_k > 0} x_0^{n_0-1} dx_0 x_1^{n_1-1} dx_1 \prod_{j=2}^k \frac{dx_j}{1 - x_j} \\ &= \frac{1}{n_0} \int_{1 > x_1 > \dots > x_k > 0} (1 - x_1^{n_0}) x_1^{n_1-1} dx_1 \prod_{j=2}^k \frac{dx_j}{1 - x_j} \\ &= \frac{1}{n_0 n_1} \int_{1 > x_2 > \dots > x_k > 0} \frac{1 - x_2^{n_1}}{1 - x_2} dx_2 \prod_{j=3}^k \frac{dx_j}{1 - x_j} \\ &\quad - \frac{1}{n_0(n_0 + n_1)} \int_{1 > x_2 > \dots > x_k > 0} \frac{1 - x_2^{n_0+n_1}}{1 - x_2} dx_2 \prod_{j=3}^k \frac{dx_j}{1 - x_j} \\ &= \frac{1}{n_0 n_1} \sum_{p_2=1}^{n_1} \frac{1}{p_2} \int_{1 > x_3 > \dots > x_k > 0} \frac{1 - x_3^{p_2}}{1 - x_3} dx_3 \prod_{j=4}^k \frac{dx_j}{1 - x_j} \\ &\quad - \frac{1}{n_0(n_0 + n_1)} \sum_{p_2=1}^{n_0+n_1} \frac{1}{p_2} \int_{1 > x_3 > \dots > x_k > 0} \frac{1 - x_3^{p_2}}{1 - x_3} dx_3 \prod_{j=4}^k \frac{dx_j}{1 - x_j} \end{aligned}$$

and so on (in the last step, we use $\frac{1-x_2^N}{1-x_2} = \sum_{p_2=1}^N x_2^{p_2-1}$). The results are

$$\begin{aligned} &\int_{1 > x_0 > \dots > x_k > 0} x_0^{n_0-1} dx_0 x_1^{n_1-1} dx_1 \prod_{j=2}^k \frac{dx_j}{1 - x_j} \\ &= \frac{1}{n_0 n_1} \sum_{n_1 \geq p_2 \geq \dots \geq p_k > 0} \frac{1}{p_2 \cdots p_k} - \frac{1}{n_0(n_0 + n_1)} \sum_{n_0+n_1 \geq p_2 \geq \dots \geq p_k > 0} \frac{1}{p_2 \cdots p_k}, \\ &\int_{1 > x_0 > \dots > x_k > 0} x_0^{n_0-1} dx_0 x_1^{n_1-1} dx_1 \prod_{j=2}^k \frac{dx_j}{1 - x_j} \\ &= \frac{1}{n_0 n_1} \sum_{n_1 \geq q_2 \geq \dots \geq q_l > 0} \frac{1}{q_2 \cdots q_l} - \frac{1}{n_0(n_0 + n_1)} \sum_{n_0+n_1 \geq q_2 \geq \dots \geq q_l > 0} \frac{1}{q_2 \cdots q_l}. \end{aligned}$$

Therefore, we get

$$\eta(1, k; 1, l) = S_1 - S_2 - S_3 + S_4,$$

where

$$\begin{aligned}
S_1 &= \sum_{n_0, n_1 > 0} \frac{1}{n_0^2 n_1^2} \sum_{\substack{n_1 \geq p_2 \geq \dots \geq p_k > 0 \\ n_1 \geq q_2 \geq \dots \geq q_l > 0}} \frac{1}{p_2 \cdots p_k q_2 \cdots q_l}, \\
S_2 &= \sum_{n_0, n_1 > 0} \frac{1}{n_0^2 n_1 (n_0 + n_1)} \sum_{\substack{n_0 + n_1 \geq p_2 \geq \dots \geq p_k > 0 \\ n_1 \geq q_2 \geq \dots \geq q_l > 0}} \frac{1}{p_2 \cdots p_k q_2 \cdots q_l}, \\
S_3 &= \sum_{n_0, n_1 > 0} \frac{1}{n_0^2 n_1 (n_0 + n_1)} \sum_{\substack{n_1 \geq p_2 \geq \dots \geq p_k > 0 \\ n_0 + n_1 \geq q_2 \geq \dots \geq q_l > 0}} \frac{1}{p_2 \cdots p_k q_2 \cdots q_l}, \\
S_4 &= \sum_{n_0, n_1 > 0} \frac{1}{n_0^2 (n_0 + n_1)^2} \sum_{\substack{n_0 + n_1 \geq p_2 \geq \dots \geq p_k > 0 \\ n_0 + n_1 \geq q_2 \geq \dots \geq q_l > 0}} \frac{1}{p_2 \cdots p_k q_2 \cdots q_l}.
\end{aligned}$$

We show that $S_1, S_2, S_3, S_4 \in \mathcal{Z}_{k+l+2}$. First, we see that $S_1 = \zeta(2)\eta(k; l)$ by (4.3), hence $S_1 \in \mathcal{Z}_2 \cdot \mathcal{Z}_{k+l} \subset \mathcal{Z}_{k+l+2}$.

Next we observe that

$$S_2 = \int_{\substack{0 < t_0 < t_1 < t_2 < 1 \\ 0 < u_k < \dots < u_1 < t_2 \\ 0 < v_l < \dots < v_2 < t_2}} \frac{dt_0}{1-t_0} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \prod_{j=1}^{k-1} \frac{du_j}{u_j(1-u_j)} \frac{du_k}{1-u_k} \prod_{j=2}^l \frac{dv_j}{1-v_j}.$$

This is proved by computing the integrals repeatedly in the order

$$u_k, \dots, u_2, u_1, t_0, t_1, t_2, v_2, \dots, v_l,$$

to which the factors of the denominator

$$p_k, \dots, p_2, n_1, n_0, n_0 + n_1, q_2, \dots, q_l$$

correspond, respectively. On the other hand, we expand

$$\frac{du_j}{u_j(1-u_j)} = \frac{du_j}{u_j} + \frac{du_j}{1-u_j}$$

for $j = 1, \dots, k-1$, and then decompose the domain of the integral into finitely many regions, in each of which the integral is of the form (3.1). Therefore, S_2 is a finite sum of multiple zeta values of weight $k+l+2$ (= the number of variables in the integral).

By symmetry, we also have $S_3 \in \mathcal{Z}_{k+l+2}$.

Finally, we note that S_4 coincides with the right hand side of (4.1) in which (r, k, l) is replaced with $(2, k+1, l+1)$. In particular, we have $S_4 \in \mathcal{Z}_{k+l+2}$. This completes the proof of the theorem. \square

A Values of $\eta(k_1, \dots, k_r; l)$ for positive integers k_1, \dots, k_r, l

The formula (4.3) for $\eta(k; l)$, which is a special case of (4.1), can also be generalized to another direction, namely, a formula for values $\eta(k_1, \dots, k_r; l)$ of the

function (1.1). Such a formula was first found by M. Kaneko as a conjecture. In this appendix, we prove it.

To state the formula, we recall the notion of the Hoffman dual \mathbf{k}^\vee of an index \mathbf{k} . This is the unique index such that $|\mathbf{k}^\vee| = |\mathbf{k}|$ and $J(\mathbf{k}^\vee) = \{1, \dots, |\mathbf{k}| - 1\} \setminus J(\mathbf{k})$.

Theorem A.1. *For a nonempty index $\mathbf{k} = (k_1, \dots, k_r)$ and an integer $l > 0$, we have*

$$\eta(\mathbf{k}; l) = (-1)^{d(\mathbf{k}^\vee)} \sum_{\mathbf{k}' \succeq \mathbf{k}^\vee} (-1)^{d(\mathbf{k}')} \zeta((\mathbf{k}')^* \circledast \underbrace{(1, \dots, 1)^*}_l). \quad (\text{A.1})$$

When $r = 1$, the Hoffman dual of (k) is $\underbrace{(1, \dots, 1)}_k$, and it has no refinement other than itself. Hence we recover (4.3) from (A.1).

For the proof of Theorem A.1, we need some preparations.

Fix positive integers k and l . In the following, indices denoted by \mathbf{k} or \mathbf{k}' are of weight k and sets denoted by J or J' are subsets of $\{1, \dots, k-1\}$. For such a set J , we put

$$S_J := \sum_{0 < a_1 \square_1 a_2 \square_2 \dots \square_{k-1} a_k = b_l \geq \dots \geq b_1 > 0} \frac{1}{a_1 \dots a_k b_l \dots b_1},$$

where \square_j for $j = 1, \dots, k-1$ denote the relational operators

$$\square_j = \begin{cases} < & (j \in J), \\ = & (j \notin J) \end{cases}$$

(it also depends on l , which we fixed). It is easy to see, for $\mathbf{k} = (k_1, \dots, k_r)$, that

$$S_{J(\mathbf{k})} = \sum_{0 < m_1 < \dots < m_r = b_l \geq \dots \geq b_1 > 0} \frac{1}{m_1^{k_1} \dots m_r^{k_r} b_1 \dots b_l} = \zeta(\mathbf{k} \circledast \underbrace{(1, \dots, 1)^*}_l). \quad (\text{A.2})$$

Lemma A.2. *For any index \mathbf{k} (of weight k), we have*

$$\xi(\mathbf{k}; l) = S_{J(\mathbf{k})}, \quad (\text{A.3})$$

$$\zeta(\mathbf{k}^* \circledast \underbrace{(1, \dots, 1)^*}_l) = \sum_{J \subset J(\mathbf{k})} S_J. \quad (\text{A.4})$$

Proof. In the definition (1.2) of $\xi(\mathbf{k}; s)$, make a change of variable $u = 1 - e^{-t}$ and substitute $s = l$. Then we have

$$\begin{aligned} & \xi(\mathbf{k}; l) \\ &= \frac{1}{(l-1)!} \int_0^1 \frac{\text{Li}_{\mathbf{k}}(u)}{u} (-\log(1-u))^{l-1} du \\ &= \sum_{0 < m_1 < \dots < m_r} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \int_0^1 u^{m_r-1} du \int_{0 < v_1 < \dots < v_{l-1} < u} \frac{dv_1}{1-v_1} \dots \frac{dv_{l-1}}{1-v_{l-1}}. \end{aligned}$$

Here we use $-\log(1-u) = \int_0^u \frac{dv}{1-v}$. The identity (A.3) is obtained by computing the integration in the order u, v_{l-1}, \dots, v_1 .

The identity (A.4) follows from the computation

$$\begin{aligned} \zeta(\mathbf{k}^* \circledast \underbrace{(1, \dots, 1)}_l) &= \sum_{\mathbf{k}' \preceq \mathbf{k}} \zeta(\mathbf{k}' \circledast \underbrace{(1, \dots, 1)}_l) \\ &= \sum_{\mathbf{k}' \preceq \mathbf{k}} S_{J(\mathbf{k}')} = \sum_{J \subset J(\mathbf{k})} S_J, \end{aligned}$$

where we use (A.2) and the correspondence between indices $\mathbf{k}' \preceq \mathbf{k}$ and sets $J \subset J(\mathbf{k})$. \square

In addition to the above lemma, we use the following identity due to Kaneko and Tsumura [3, Proposition 3.2]:

$$\eta(\mathbf{k}; s) = (-1)^{d(\mathbf{k})-1} \sum_{\mathbf{k}' \succeq \mathbf{k}} \xi(\mathbf{k}'; s). \quad (\text{A.5})$$

Proof of Theorem A.1. Let us compute the sum in the right hand side of (A.1):

$$\begin{aligned} \sum_{\mathbf{k}' \succeq \mathbf{k}^\vee} (-1)^{d(\mathbf{k}')} \zeta(\underbrace{(\mathbf{k}')^* \circledast (1, \dots, 1)}_l) &= \sum_{\mathbf{k}' \succeq \mathbf{k}^\vee} (-1)^{d(\mathbf{k}')} \sum_{J \subset J(\mathbf{k}')} S_J \\ &= \sum_{J' \supset J(\mathbf{k}^\vee)} (-1)^{\#J'+1} \sum_{J \subset J'} S_J \\ &= \sum_J S_J \sum_{J' \supset J(\mathbf{k}^\vee) \cup J} (-1)^{\#J'+1}. \end{aligned}$$

Here, we use (A.4) in the first step and the correspondence between indices $\mathbf{k}' \succeq \mathbf{k}^\vee$ and sets $J' \supset J(\mathbf{k}^\vee)$ in the second step (note that $d(\mathbf{k}') = \#J(\mathbf{k}') + 1$). The third step is just an exchange of summations.

It is easily shown that

$$\sum_{J' \supset J(\mathbf{k}^\vee) \cup J} (-1)^{\#J'+1} = \begin{cases} 0 & (J(\mathbf{k}^\vee) \cup J \subsetneq \{1, \dots, k-1\}), \\ (-1)^k & (J(\mathbf{k}^\vee) \cup J = \{1, \dots, k-1\}). \end{cases}$$

Because of the equivalence

$$J(\mathbf{k}^\vee) \cup J = \{1, \dots, k-1\} \iff J \supset \{1, \dots, k-1\} \setminus J(\mathbf{k}^\vee) = J(\mathbf{k}),$$

we can continue the above computation as

$$\begin{aligned} \sum_J S_J \sum_{J' \supset J(\mathbf{k}^\vee) \cup J} (-1)^{\#J'+1} &= (-1)^k \sum_{J \supset J(\mathbf{k})} S_J \\ &= (-1)^k \sum_{\mathbf{k}' \succeq \mathbf{k}} \xi(\mathbf{k}'; l) \\ &= (-1)^k (-1)^{d(\mathbf{k})-1} \eta(\mathbf{k}; l), \end{aligned}$$

using (A.3) and (A.5). Thus the desired identity (A.1) is obtained if we notice that $k-1 = \#J(\mathbf{k}) + \#J(\mathbf{k}^\vee) = (d(\mathbf{k})-1) + (d(\mathbf{k}^\vee)-1)$, i.e., $k - (d(\mathbf{k})-1) = d(\mathbf{k}^\vee)$. \square

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